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SKEWNESS AND NON-GAUSSIAN INITIAL CONDITIONS

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ABSTRACT

We compute the skewness of the galaxy distribution arising from the nonlinear evolution of arbitrary non-Gaussian initial conditions to second order in perturbation theory including the effects of nonlinear biasing. The result contains a term which is identical to that for a Gaussian initial distribution plus terms which depend on the skewness and kurtosis of the initial conditions. The results are model dependent; we present calculations for several toy models. At late times, the leading contribution from initial skewness decays away and becomes increasingly unimportant, but the contribution from initial kurtosis, previously overlooked, has the same time dependence as the Gaussian terms. Observations of a linear dependence of the normalized skewness on the rms density fluctuation therefore do not necessarily rule out initially non-Gaussian models. Although initial conditions could in principle have negative skewness and kurtosis, we present bounds that show that to second order in perturbation theory, skewness necessarily grows more positive with time.

Subject Headings: galaxies: clustering — large-scale structure of the universe

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1. Introduction

It is often assumed that the primordial fluctuations that generated large-scale cosmological structure were effectively Gaussian. This simplification is not without some justification; the simplest realizations of inflation do result in Gaussian fluctuations in the inflating scalar field, that are then imparted to the matter. Small departures of the primordial distribution from pure Gaussian may be unimportant: nonlinear gravitational evolution induces a skewness that grows with time relative to the linearly evolved initial skewness, and the observed non-Gaussian nature of the distribution of galaxies on small scales may be entirely the result of nonlinear gravitational clustering. It has certainly been possible to generate a wide range of "present-day" universes in numerical simulations starting from Gaussian initial conditions, and there are indications in the simulations that the initial power spectrum is forgotten on scales that have evolved into the strongly nonlinear regime. The popularity of Gaussian models is not diminished by the possibilities they present for exact calculations.

However, the source of density fluctuations is at present an open question, and interesting models such as cosmic strings and global texture do provide fluctuations which are non-Gaussian initially. As observations reach to larger and larger scales, it is becoming possible to measure statistical properties of structure that are more closely related to conditions in the very early universe, and it may be possible to determine by measurement whether the primordial fluctuations were in fact Gaussian or not. One quantitative measure of departures from a Gaussian is the third moment of the density contrast, or skewness. A number of analytic and numerical investigations of the evolution of skewness have been undertaken (Peebles 1980; Coles & Frenk 1991; Bouchet *et al.* 1992; Juszkiewicz & Bouchet 1992; Lahav *et al.* 1992; Weinberg & Cole 1992; Luo & Schramm 1993; Juszkiewicz *et al.* 1993; Coles *et al.* 1993). Measurements of the volume-averaged skewness have been given for the QDOT survey (Saunders *et al.* 1991; Coles & Frenk 1991) and the 1.2 Jy IRAS survey (Bouchet *et al.* 1993).

In this paper we compute the evolution of skewness in second-order perturbation theory for arbitrary non-Gaussian initial density fields. A similar calculation for some special cases has been done by Luo & Schramm (1993). The calculation breaks naturally into successive stages. In § 2.1 we introduce second order perturbation theory results from gravitational instability. In § 2.2 we evaluate the induced skewness for Gaussian initial conditions; in § 2.3 we compute the additional terms that arise from non-Gaussian initial conditions, including a term from the initial four-point function previously overlooked, and in § 2.4 we show the further effects of nonlinear bias. We find that all three sources of skewness (nonlinear contributions from Gaussian initial conditions, initial non-Gaussian correlations, and nonlinear bias) can contribute terms of comparable amplitude and identical time dependence to the skewness of the galaxy distribution, and thus all must be considered in interpreting observations.

2. Gravity and Skewness

2.1. Perturbation Theory

We study fluctuations in the cosmological mass density, $\delta(\boldsymbol{x},t) = [\rho(\boldsymbol{x},t) - \bar{\rho}]/\bar{\rho}$, and in Section 3 below in the possibly different galaxy number density $\delta_g = \delta n_g/\bar{n}_g$. Statistical properties can be summarized in the n-point moments, $\langle \delta^n \rangle$, or their irreducible reductions ξ_n. By construction, $\langle \delta(\boldsymbol{x}) \rangle = 0$, and the next few moments are

$$\begin{aligned}\langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2) \rangle &= \xi_{12},\\ \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\delta(\boldsymbol{x}_3) \rangle &= \zeta_{123},\\ \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\delta(\boldsymbol{x}_3)\delta(\boldsymbol{x}_4) \rangle &= \xi_{12}\,\xi_{34} + \xi_{13}\,\xi_{24} + \xi_{23}\,\xi_{14} + \eta_{1234}.\end{aligned} \tag{1}$$

For a homogeneous and isotropic density distribution, the irreducible two-, three-, and four-point functions ξ_{12}, ζ_{123}, and η_{1234} depend only on relative distances, $\xi_{12} = \xi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$, *etc.*, and not on absolute positions and orientations.

To linear order in perturbation theory, the density contrast grows with time by an overall scale factor, $\delta(\boldsymbol{x},t) = A(t)\delta(\boldsymbol{x},t_0)$, where $\delta(\boldsymbol{x},t_0)$ is the primordial seed fluctuation at an early initial time t_0 and $A(t_0) \equiv 1$. For the simplest case (matter dominated, $\Omega = 1$, $k = \Lambda = 0$), $A(t) \propto a(t)$, the cosmological expansion factor, and appropriate normalizations can be chosen such that $A(t) = a(t)$. In linear theory, initially Gaussian fluctuations remain Gaussian: the distribution has no irreducible moments beyond the two-point function, and in particular has vanishing skewness, $\zeta = 0$.

For nonlinear fluctuations, in perturbation theory we expect to generate a series with terms of all orders arising from couplings between modes,

$$\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots, \tag{2}$$

where $\delta^{(n)} \sim \mathcal{O}(\delta_0^n)$ (see Goroff *et al.* 1986). These higher order terms induce nonvanishing higher order irreducible moments; terms up to order $\delta^{(n-1)}$ are required to compute ξ_n. Including the first correction, the three-point function is

$$\zeta = \langle [\delta^{(1)} + \delta^{(2)} + \cdots]^3 \rangle = \langle [\delta^{(1)}]^3 \rangle + 3\,\langle [\delta^{(1)}]^2\delta^{(2)} \rangle + \cdots. \tag{3}$$

For Gaussian initial conditions, the $\langle [\delta^{(1)}]^3 \rangle$ term vanishes, but in general the next term $\langle [\delta^{(1)}]^2\delta^{(2)} \rangle$ does not.

In gravitational instability, the second order contribution $\delta^{(2)}$ is (Peebles 1980; Fry 1984; Goroff *et al.* 1986)

$$\delta^{(2)} = \frac{5}{7}\delta^2 - \delta_{,i}\Delta_{,i} + \frac{2}{7}\Delta_{,ij}\Delta_{,ij}, \tag{4}$$

where

$$\Delta(\boldsymbol{x}) = \int \frac{d^3x'}{4\pi}\frac{\delta(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{5}$$

In this and following, expressions on the right-hand side are to be evaluated for the linearly evolved initial distribution: $\delta = a(t)\delta(\boldsymbol{x}, t_0)$. Throughout, zero subscripts denote quantities derived from linear evolution of the initial density field, $\xi_0 = a^2(t)\xi(x, t_0)$, $\zeta_0 = a^3(t)\zeta(\boldsymbol{x}_i, t_0)$, *etc.* Thus, including the first correction in equation (3), the third moment becomes

$$\zeta = \left\langle \delta^3 + \frac{15}{7}\delta^4 - 3\delta^2\delta_{,i}\Delta_{,i} + \frac{6}{7}\delta^2\Delta_{,ij}\Delta_{,ij} \right\rangle, \tag{6}$$

or, with equation (5),

$$\begin{aligned}\zeta = \zeta_0(0) + \Big\langle \frac{15}{7}\delta^4(\boldsymbol{x}) - 3\int \frac{d^3x'}{4\pi}\delta^2(\boldsymbol{x})\delta_{,i}(\boldsymbol{x})\delta(\boldsymbol{x}')\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}{}_{,i} \\ + \frac{6}{7}\int \frac{d^3x d\, d^3x''}{4\pi \quad 4\pi}\delta^2(\boldsymbol{x})\delta(\boldsymbol{x}')\delta(\boldsymbol{x}'')\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}{}_{,ij}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}''|}{}_{,ij}\Big\rangle\end{aligned} \tag{7}$$

2.2. Skewness for a Gaussian Initial Distribution

For a Gaussian initial distribution, equation (7) reduces to a simple expression. All higher order irreducible moments vanish, so that $\zeta_0 = 0$, and from equation (1) with $\eta_0 = 0$ we have for the fourth moments

$$\begin{aligned}\langle \delta^4(\boldsymbol{x}) \rangle &= 3\,\xi^2(0), \\ \langle \delta^2(\boldsymbol{x})\delta_{,i}(\boldsymbol{x})\delta(\boldsymbol{x}') \rangle &= \frac{1}{3}\langle \delta^3(\boldsymbol{x})\delta(\boldsymbol{x}') \rangle_{,i} = \xi(0)\xi(\boldsymbol{x}'-\boldsymbol{x})_{,i} \\ \langle \delta^2(\boldsymbol{x})\delta(\boldsymbol{x}')\delta(\boldsymbol{x}'') \rangle &= \xi(0)\xi(\boldsymbol{x}'-\boldsymbol{x}'') + 2\,\xi(\boldsymbol{x}'-\boldsymbol{x})\xi(\boldsymbol{x}''-\boldsymbol{x})\end{aligned} \tag{8}$$

Applying $\nabla_i F(\boldsymbol{x}'-\boldsymbol{x}) = -\nabla'_i F(\boldsymbol{x}'-\boldsymbol{x})$, and translating the origin of integration leads to

$$\begin{aligned}\zeta = \frac{15}{7}[3\,\xi(0)^2] - 3\int \frac{d^3x'}{4\pi}\xi(0)\xi(x')_{,i}\frac{1}{|x'|}{}_{,i} \\ + \frac{6}{7}\int \frac{d^3x'\, d^3x''}{4\pi \quad 4\pi}\left[\xi(0)\xi(\boldsymbol{x}'-\boldsymbol{x}'') + 2\,\xi(x')\xi(x'')\right]\frac{1}{|x'|}{}_{,ij}\frac{1}{|x''|}{}_{,ij}\end{aligned} \tag{9}$$

Integrated by parts, the second term becomes $-\xi^2(0)$. In the double integral, the term containing $\xi(\boldsymbol{x}'-\boldsymbol{x}'')$, also integrated by parts, gives $\xi^2(0)$, while, using the identity

$$\nabla_i\nabla_j\frac{1}{|\boldsymbol{x}|} = \frac{3\hat{x}_i\hat{x}_j - \delta_{ij}}{x^3} - \frac{4\pi}{3}\delta_{ij}\delta_{\rm D}(\boldsymbol{x}) \tag{10}$$

($\delta_{\rm D}$ is the Dirac δ-function), the term containing $\xi(x')\xi(x'')$ becomes $2\,\xi^2(0)/3$. Thus, in total, for Gaussian initial conditions

$$\zeta_G = \frac{34}{7}\xi_0^2(0), \tag{11}$$

a result first obtained by Peebles (1980). For future use, we note that combining the result obtained by integration by parts with the result implied by using equation (10) gives

$$I\,[\,\xi\,] = \int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi}\,\frac{6\,P_2(\hat{x}'\cdot\hat{x}'')}{x'^3x''^3}\,\xi(x'-x'') = \frac{2}{3}\,\xi(0)\,. \tag{12}$$

In interpreting skewness, there are two reduced or normalized ratios that are useful to consider. First, deviations from a Gaussian can be characterized by the ratios $\lambda_n = \xi_n/\sigma^n$, where $\sigma^2 = \xi(0)$. For a distribution to be nearly Gaussian, the λ_n must be small. For the third moment, the normalized skewness $s \equiv \lambda_3$ is

$$s_G = \frac{\zeta}{\sigma^3} = \frac{34}{7}\sigma(t). \tag{13}$$

When clustering is weak, $\xi(0) \ll 1$, then $s \ll 1$ also, and the distribution is indeed almost Gaussian; but, since σ grows as $a(t)$, even for an initial Gaussian, departures from Gaussian increase with time. Also commonly seen are the hierarchical amplitudes S_n, where $\xi_n = S_n\,\xi^{n-1}$. For any nonlinear transformation of a weakly-correlated Gaussian, the leading contribution to ξ_n is of the hierarchical form with S_n constant (see Fry & Gaztañaga 1993). For the case of Gaussian initial conditions, we see that gravitational instability gives

$$S_{3,G} = \frac{\zeta}{\xi^2} = \frac{34}{7}. \tag{14}$$

Non-Gaussian initial conditions and nonlinear bias modify these simple results considerably.

2.3. Skewness for a Non-Gaussian Initial Distribution

For non-Gaussian initial conditions there are additional contributions to the skewness. The leading term $\langle\,\delta^3\,\rangle = \zeta(0)$ no longer vanishes, and there is also an additional contribution to $\langle\,[\delta^{(1)}]^2\delta^{(2)}\,\rangle$ from the four-point function. The first term is $15\,\eta(0)/7$; the second can again be integrated by parts, and gives $-\eta(0)$. Using the translational invariance of η in the final term, we arrive at

$$\zeta = \zeta_0(0) + \frac{34}{7}\xi_0^2(0) + \frac{8}{7}\eta_0(0) + \frac{6}{7}\int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi}\,\eta_0(0,0,x',x'')\frac{1}{|x'|}{}_{,ij}\frac{1}{|x''|}{}_{,ij} \tag{15}$$

or, using equation (10), the alternative form

$$\zeta = \zeta_0(0) + \frac{34}{7}\,\xi_0^2(0) + \frac{10}{7}\,\eta_0(0) + \frac{6}{7}\,I\,[\,\eta_0\,] \tag{16}$$

where

$$I\,[\,\eta\,] = \int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi}\frac{6\,P_2(\hat{x}'\cdot\hat{x}'')}{x'^3x''^3}\,\eta(0,0,x',x''). \tag{17}$$

The difference between the complete reduction achieved in equation (11) and the inability to do so for the non-Gaussian case can be traced to the difference between the connected and disconnected parts of $\langle \delta^4 \rangle$. The Gaussian contribution to $\langle \delta^2(x)\delta(x')\delta(x'') \rangle$ separates into a term that depends only on $\xi(x'-x'')$ and a term that depends only on $\xi(x')\xi(x'')$, both of which can be integrated, so that in equation (12), $I[\xi] = 2\xi(0)/3$. Certainly we expect that $\eta(0)$ gives an order-of-magnitude estimate of $I[\eta]$, but for the connected four-point function $\eta(0,0,x',x'')$, in general $I[\eta]$ does not evaluate simply.

Without repeating the details, we note that a similar calculation gives a correction to the two-point function also, $\langle \delta^2 \rangle = \langle [\delta^{(1)}]^2 \rangle + 2\langle [\delta^{(1)}]^2 \delta^{(2)} \rangle$, which for gravitational instability is

$$\xi = \xi_0(0) + \frac{13}{21}\zeta_0(0) + \frac{4}{7} I[\zeta_0], \tag{18}$$

where $I[\zeta]$ is the integral analogous to equation (12) or equation (17) evaluated for $\zeta(0,x',x'')$. From equation (16) and equation (18), we can write s as

$$s = s_0 + \left(\frac{34}{7} + \frac{10}{7}\frac{\eta_0}{\xi_0^2} + \frac{6}{7}\frac{I[\eta_0]}{\xi_0^2} - \frac{13}{14}\frac{\zeta_0^2}{\xi_0^3} - \frac{6}{7}\frac{\zeta_0 I[\zeta_0]}{\xi_0^3} \right) \sigma(t) \tag{19}$$

where the factor inside parentheses is independent of time. Thus the second term grows as $\sigma(t) \propto a(t)$. The calculation holds for $\xi \ll 1$ and $\xi_n \sim \sigma^n$. The hierarchical amplitude S_3 is

$$S_3 = \frac{S_{3,0}}{a(t)} + \frac{34}{7} + \frac{10}{7}\frac{\eta_0}{\xi_0^2} + \frac{6}{7}\frac{I[\eta_0]}{\xi_0^2} - \frac{26}{21}\frac{\zeta_0^2}{\xi_0^3} - \frac{8}{7}\frac{\zeta_0 I[\zeta_0]}{\xi_0^3}. \tag{20}$$

The integral $I[\eta]$ and thus the complete answer depends on the details of the four-point function, as can be seen from a few "toy" models. Consider first the case of a white-noise non-Gaussian model, *i.e.*, a model in which the one-point density distribution $p(\delta)$ is non-Gaussian, but in which there are no spatial correlations. This class of models is not physically realistic, but it provides a simple set of non-Gaussian models with which to explore the dependence of the evolution of the density on the initial form of $p(\delta)$. Messina *et al.* (1990) have investigated a number of such models numerically (see also Weinberg & Cole 1992). For these models, $I[\eta] = 0$, and we obtain

$$\zeta = \zeta_0(0) + \frac{34}{7}\xi_0^2(0) + \frac{10}{7}\eta_0(0). \tag{21}$$

The requirements that the four-point function be symmetric and connected suggest more realistic models built from two-point functions, one with the minimal three connections of a tree, η_T, one where the points are connected in a closed loop, η_L, and one which links all six possible pairs of points, η_A:

$$\begin{aligned} \eta_T &= \xi_{12}\,\xi_{23}\,\xi_{34} + (\text{sym.}) && \text{(16 total terms)}, \\ \eta_L &= \xi_{12}\,\xi_{23}\,\xi_{34}\,\xi_{41} + (\text{sym.}) && \text{(3 total terms)}, \\ \eta_A &= \xi_{12}\,\xi_{13}\,\xi_{14}\,\xi_{23}\,\xi_{24}\,\xi_{34}. \end{aligned} \tag{22}$$

For $\eta(0,0,x',x'')$, these models give

$$
\begin{aligned}
\eta_T &= 4\,\xi(0)\xi'\xi'' + 2\,\xi'\xi''(\xi' + \xi'') + \xi\,[2\xi(0)(\xi' + \xi'') + (\xi' + \xi'')^2], \\
\eta_L &= \xi'^2\xi''^2 + 2\xi(0)\xi'\xi''\xi, \\
\eta_A &= \xi(0)\xi'^2\xi''^2\xi,
\end{aligned} \tag{23}
$$

where $\xi' = \xi(x')$, $\xi'' = \xi(x'')$, and $\xi = \xi(|x' - x''|)$. In computing the integral over angles in equation (16), terms in η that are isotropic, *i.e.*, those that do not contain ξ, vanish, as did analogous terms in equation (9). Since $\xi(x) < \xi(0)$, the tree model gives

$$
\begin{aligned}
I\,[\eta_T] &= \int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi}\frac{6P_2(\hat{x}'\cdot\hat{x}'')}{x'^3x''^3}\xi\,[2\,\xi(0)(\xi' + \xi'') + \xi'^2 + \xi''^2 + 2\,\xi'\xi''] \\
&\leq \int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi}\frac{6P_2(\hat{x}'\cdot\hat{x}'')}{x'^3x''^3}\xi\,[8\,\xi^2(0)] = \frac{2}{3}\,\xi(0)\,[8\,\xi^2(0)] = \frac{1}{3}\,\eta_T(0);
\end{aligned} \tag{24}
$$

similarly, the other models give upper bounds $4\,\eta_L(0)/9$ and $2\,\eta_A(0)/3$. Numerical investigation of these toy models shows that the numerical value of the integral depends on the functional dependence of $\xi(x)$ as well as the particular model of the four-point function, and is usually considerably smaller than these conservative upper limits. Table 1 lists some numerical results.

As a second class of interesting non-Gaussian models, we consider the seed models examined by Scherrer & Bertschinger (1991) and by Scherrer (1992), in which the density field is the convolution of a fixed density profile or set of density profiles with a distribution of points. The global texture model (Gooding *et al.* 1992 and references therein), for example, can be described in this way. Here we will consider only a very simple class of seed models, in which the seeds are identical and randomly distributed, so that the density perturbation at $\boldsymbol{x}$ is

$$
\delta(\boldsymbol{x}) = \sum_i f(\boldsymbol{x} - \boldsymbol{x}_i), \tag{25}
$$

where $f(\boldsymbol{x})$ is the accretion pattern around a single seed. This is a special case of the more general models examined by Scherrer & Bertschinger (1991) and by Scherrer (1992). For such a model (Scherrer & Bertschinger 1991) it is easy to show that

$$
\xi_n(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = n_0 \int d^3x\, f(\boldsymbol{x}_1 - \boldsymbol{x})\, f(\boldsymbol{x}_2 - \boldsymbol{x}) \cdots f(\boldsymbol{x}_N - \boldsymbol{x}), \tag{26}
$$

where n_0 is the number density of the seeds. This equation is valid when f satisfies the integral constraint

$$
\int d^3x\, f(\boldsymbol{x}) = 0, \tag{27}
$$

which insures that $\langle\, \delta \,\rangle = 0$. For these seed models, evaluation of the last term in equation (16) can be difficult, and it is easier here to work from equation (15). In this equation, using equation (26), the last term becomes

$$
\begin{aligned}
&\int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi}\eta(0, 0, \boldsymbol{x}', \boldsymbol{x}'')\frac{1}{|\boldsymbol{x}'|}_{,ij}\frac{1}{|\boldsymbol{x}''|}_{,ij} \\
&\quad = n_0 \int \frac{d^3x'}{4\pi}\frac{d^3x''}{4\pi} d^3x\, f^2(\boldsymbol{x}) f(\boldsymbol{x}') f(\boldsymbol{x}'')\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}_{,ij}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}''|}_{,ij},
\end{aligned} \tag{28}
$$

again valid when the seeds are randomly distributed. As an example, consider a simple model where the density field around a seed is a positive spherical top hat of radius r_1 embedded in a negative spherical tophat of radius r_2, with $r_2 > r_1$. Specifically, we take

$$f(\boldsymbol{x}) = \begin{cases} f_0 - g_0, & r < r_1, \\ \quad - g_0, & r_1 < r < r_2 \end{cases} \tag{29}$$

($r = |\boldsymbol{x}|$); the integral constraint given by equation (27) is satisfied by requiring that $f_0 V_1 = g_0 V_2$, where V_1 and V_2 are the volumes of spheres of radius r_1 and r_2, respectively. Finally, we define the free parameter t to be given by $t = V_1/V_2 = g_0/f_0$, so that $0 < t < 1$. Then equation (26) gives:

$$\xi_n(0) = n_0 V_2\, g_0^n \left[t\,(\frac{1}{t} - 1)^n + (-1)^n (1-t) \right], \tag{30}$$

and equation (28) gives:

$$\int \frac{d^3x'}{4\pi} \frac{d^3x''}{4\pi} \eta(0,0,\boldsymbol{x}',\boldsymbol{x}'') \frac{1}{|x'|}_{,ij} \frac{1}{|x''|}_{,ij} = n_0 V_2\, g_0^4 \left[\frac{1}{3} \left(\frac{1}{t} - 1 \right) \left(\frac{1}{t^2} - \frac{3}{t} + 5 \right) \right]. \tag{31}$$

For comparison with the other models above, the integral $I\,[\,\eta\,]$ here is

$$I\,[\,\eta\,] = \frac{2}{3} \frac{t^2}{(1-t)^3 + t^3}\, \eta(0) \tag{32}$$

The factor on the right hand side varies from 0 for $t = 0$ to a maximum of 8/9 for $t = 2/3$ to 2/3 for $t = 1$.

2.4. Bias and Skewness

There is finally the additional possibility that the distribution of galaxies is not the same as that of the underlying mass density, that the locations where galaxies form are "biased." If the number density of galaxies is uniquely determined by the mass density, then we can write $n(\boldsymbol{x})$ as a functional, $n(\boldsymbol{x}) = F[\rho(\boldsymbol{x})]$. As a simpler model, assume that $n = f(\rho)$, a simple function of the local density, and that near $\rho = \bar{\rho}$ we can write n_g as a power series in ρ, $n_g = \sum a_k \rho^k$, or, equivalently δ_g as a series in δ_ρ,

$$\delta_g = \sum_{k=0}^{\infty} b_k \delta_\rho^k \tag{33}$$

(Fry & Gaztañaga 1993). The constant term b_0 is chosen so that $\langle\, \delta_g \,\rangle = 0$, and the usual linear bias parameter is $b = b_1$. It is easy to compute the connected second and third moments of the biased distribution in terms of the underlying mass distribution,

$$\begin{aligned} \xi_g &= \langle\, \delta_g^2 \,\rangle = b^2 \xi_\rho + 2 b b_2 \zeta_\rho, \\ \zeta_g &= \langle\, \delta_g^3 \,\rangle = b^3 \zeta_\rho + 6 b^2 b_2 \xi_\rho^2 + 3 b^2 b_2 \eta_\rho, \end{aligned} \tag{34}$$

where ξ_ρ and ζ_ρ are given by equations (18) and (16) and we have again dropped higher-order terms in both equations.

Including the bias contributions to the normalized skewness s and the hierarchical amplitude S_3, we obtain

$$s = s_0 + \left(\frac{34}{7} + \frac{6b_2}{b^2} + \left[\frac{10}{7} + \frac{3b_2}{b} \right] \frac{\eta_0}{\xi_0^2} + \frac{6}{7} \frac{I[\eta_0]}{\xi_0^2} - \left[\frac{13}{14} + \frac{3b_2}{b} \right] \frac{\zeta_0^2}{\xi_0^3} - \frac{6}{7} \frac{\zeta_0 I[\zeta_0]}{\xi_0^3} \right) \sigma(t) \tag{35}$$

$$S_3 = \frac{S_{3,0}}{b} \frac{1}{a(t)} + \frac{1}{b} \frac{34}{7} + \frac{6b_2}{b^2} + \left(\frac{10}{7b} + \frac{3b_2}{b^2} \right) \frac{\eta_0}{\xi_0^2} + \frac{6}{7b} \frac{I[\eta_0]}{\xi_0^2} - \left(\frac{26}{21b} + \frac{4b_2}{b^2} \right) \frac{\zeta_0^2}{\xi_0^3} - \frac{8}{7b} \frac{\zeta_0 I[\zeta_0]}{\xi_0^3} \tag{36}$$

At late times, the first term becomes increasingly less important, but the other contributions from ζ_0 and η_0 remain of the same relative importance for all time. For the models of equation (22), these are small (and, strictly, if they are included, so should other higher order terms), but for the seeded models, with $n_0 V_2 \lesssim 1$ and for any value of g_0, these contributions are of order 1. For bias parameters b_k of order 1, the bias contribution is also of the same order as the 34/7 from gravitational instability.

3. Discussion

Our main results, equation (16) and (34) for ζ, equation (35) for s, or equation (36) for S_3, indicate that there are three significant contributions to the evolved skewness for non-Gaussian initial conditions. First, there is a term which is simply the linearly evolved initial three-point function or skewness, $\zeta_0 \sim a^3(t)$. Secondly, we have the usual nonlinear "Gaussian" contribution $(34/7)\xi_0^2 \sim a^4(t)$. Finally, there is a contribution, also proportional to a^4, that depends on the four-point correlation (kurtosis) of the initial density field, $\eta_0 \sim a^4(t)$, a term which has previously been overlooked. In general, the initial skewness of the density field need not dominate the skewness at late times; in equation (35), if $s_0 \lesssim 5$ and the four-point contribution is nonnegative, then the time-dependent contribution will dominate the linearly evolved initial skewness before perturbation theory breaks down. Alternatively, if s_0 and the initial four-point function are sufficiently small, then the evolved skewness will be indistinguishable from that for Gaussian initial conditions.

In the observations, both Coles & Frenk (1991) and Bouchet *et al.* (1993) find that skewness in the distribution of IRAS galaxies is consistent with gravitational evolution from Gaussian initial conditions. Our results are not directly applicable to these observations, since observers measure volume-averaged two- and three-point functions, while our calculations strictly apply to properties of density fields measured at a point. However, the volume averaging is a linear filtering that, while it may modify the result by factors of order unity, cannot add or remove terms:

$$\delta_w(\boldsymbol{x}) = \int d^3x' \, W(x') \, \delta(\boldsymbol{x} - \boldsymbol{x}'),$$
$$\xi_w(\boldsymbol{x}_i) = \int d^3x_1' d^3x_2' \, W(x_1')W(x_2') \, \xi(\boldsymbol{x}_i - \boldsymbol{x}_i'),$$
$$\zeta_w(x_1, x_2, x_3) = \int d^3x_1' d^3x_3' d^3x_3' \, W(x_1')W(x_2')W(x_3') \, \zeta(\boldsymbol{x}_i - \boldsymbol{x}_i'), \tag{37}$$

where $W(x)$ is a suitably normalized windowing function. Juszkiewicz *et al.* (1993) have calculated the skewness of the smoothed field for the Gaussian case, and they indeed find that the results are qualitatively similar to what is seen for an unsmoothed field: ζ remains proportional to $\xi_0^2(0)$, but the constant of proportionality depends on the assumed power spectrum. The calculation is rather involved, even for the Gaussian case, and the results are model-specific, depending on the particular window function (top-hat, Gaussian ball), on the initial power spectrum, and for our case on the initial three- and four-point functions, so we have not attempted to extend it to general non-Gaussian models, leaving that for specific model proponents. However, if our results for the unsmoothed density field are qualitatively similar to the skewness evolution for the smoothed field, then we can make some general comments about the sort of constraints which observations of skewness place on deviations from Gaussianity in the initial conditions. The observations do not directly constrain the initial skewness s_0, since the contribution from s_0 evolves away, as shown in equation (19). Rather, the observations bound the rather messy expression shown in this equation. The observation that s depends linearly on σ, as it appears to do in the IRAS data examined by Bouchet *et al.* (1993) is consistent with non-Gaussian models as long as the second term in equation (19) or (35) dominates the first. For non-Gaussian models which have zero initial skewness but nonzero kurtosis (i.e., models with a symmetric density distribution), s will evolve exactly proportional to σ; the only effect of the non-Gaussianity on the skewness will be the effect of η_0 on the constant of proportionality as shown in equation (19). Some of these conclusions have been noted by Luo & Schramm (1993) for several special cases.

Finally, we finish with some instructive lower bounds. The last term in equation (6) is positive, so that

$$\zeta \geq \zeta_0(0) + \frac{8}{7}[3\,\xi_0^2(0) + \eta_0(0)] \tag{38}$$

From the Schwarz inequality, $\langle \delta^4 \rangle \geq \langle \delta^2 \rangle^2$, so that $3\,\xi^2(0) + \eta(0) \geq \xi^2(0)$, we further obtain

$$\zeta \geq \zeta_0(0) + \frac{8}{7}\xi_0^2(0) \tag{39}$$

Although equation (38) is the stronger bound, equation (39) is useful, because it shows that even for density fields with $\eta < 0$, the contribution from the four-point function can never be so negative as to cause ζ to decrease with time. Thus, in second-order perturbation theory, ζ increases with time for all initial density fields. This agrees with physical intuition:

gravitational clustering leads to a small number of high density regions and a relatively larger volume of low density regions, regardless of initial conditions.

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Table 1

Numerical Values for $I[\eta]/\eta(0)$

$\xi(x)$	$I[\xi]$	$I[\eta_T]$	$I[\eta_L]$	$I[\eta_A]$
$1/(1+x)$	0.667	0.121	0.122	0.101
$\exp(-x)$	0.667	0.114	0.111	0.093
$1/(1+x^2)$	0.667	0.104	0.094	0.065
$\exp(-x^2)$	0.667	0.092	0.072	0.046

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